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LETTER TO THE EDITOR

Field theories for kinetic growth models

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Abstract. Field theories for kinetic growth models, such as DLA or the Eden model, are formulated using simple reaction-diffusion master equations. Our approach, based on Poisson transform techniques, allows us to specify the stochastic fluctuations *exactly*. We are therefore, for the first time in an ideal position to study mean-field approximations, fluctuations and possibly the renormalisation group.

Models for irreversible kinetic growth such as the Eden model (Eden 1961) and the diffusion limited aggregation model (DLA) (Witten and Sander 1981, 1983) have in recent years attracted a great deal of attention. Physical realisations of these and other intrinsically irreversible processes arise in many important areas, notable amongst which are reaction-diffusion models (Schlögl 1972, Elderfield and Vvedensky 1985a), biological pattern formation (Meinhardt 1982), epidemic processes with immunisation (Cardy 1983) and automata (Wolfram 1983). In this letter we show how, starting from a reaction-diffusion master equation for the Eden model and DLA, one can derive an *exact* field theoretical representation. Given such a description, one is then for the first time in an ideal position to study mean-field approximations, fluctuations and possibly the renormalisation group and scaling. Our analysis complements, clarifies and extends the phenomenological approach of Parisi and Zhang (1985, PZ), who used phenomenological equations for the density and first correlation function to infer an underlying stochastic dynamics. At the mean-field (or deterministic) level of approximation the two approaches lead to qualitatively similar predictions; however, there are important differences when one considers the fluctuation corrections. Our field theory, being an *exact* representation of a physically motivated master equation for the 'microscopic' dynamics, would therefore seem to be a better starting point for the systematic study of irreversible kinetic growth at macroscopic scales.

Let us first consider the Eden model. Briefly, this model describes a process in which an introduced seed grows on a lattice by creating particles at nearby sites, which then in turn grow. Steps which would lead to a multiply occupied site are excluded. Defining x_i to be the number of particles at the site i, one is therefore led to consider the reaction scheme

$$X_i \xrightarrow{k} X_i + X_j \tag{1}$$

$$2X_i \xrightarrow{e} X_i. \tag{2}$$

Here the first step describes the growth process (i, j nearest neighbours) whilst the second quickly corrects errors due to multiple occupancy ($e \gg k$). Associated with this reaction is a master equation

$$\frac{\partial}{\partial t}P(\{x_i\},t)=k\sum_{\substack{i,j=1\\ \mathrm{NN}}}^{N}x_j(P(x_1,\ldots,x_i-1,\ldots,x_N,t)-P(\{x_i\},t))$$
$$+e\sum_{i=1}^{N}x_i(x_i+1)P(x_1,\ldots,x_i+1,\ldots,x_N,t)-x_i(x_i-1)P(\{x_i\},t) \qquad (3)$$

where $P(x_1,x_2,\ldots,x_N,t)$ is the probability of finding $\{x_i\}$ particles at time t. As an initial condition we have

$$P(x_1,x_2\ldots x_N,0)=\delta(x_1-1)\prod_{i=2}^{N}\delta(x_i). \qquad (4)$$

To derive in a simple manner the field theoretical representation for the Eden model it is helpful to first transform the master equation (3) into a Fokker-Planck form using a Poisson transformation (Gardiner and Chaturvedi 1977, Gardiner 1983, Elderfield 1985a). Introducing a quasi-probability $f(\{\alpha_i\},t)$ via

$$P(\{x_i\},t)\equiv\int_{\mathscr{D}}\prod_{i=1}^{N}\left(\frac{\mathrm{d}\mu(\alpha_i)\,\mathrm{e}^{-\alpha_i}(\alpha)^{x_i}}{(x_i)!}\right)f(\{\alpha_i\},t) \qquad (5)$$

where for simplicity we consider only the real Poisson representation ($\mathrm{d}\mu(\alpha)=\mathrm{d}\alpha$, $\mathscr{D}\subset\mathscr{R}$), one finds on integrating by parts that (3) may be rewritten in the Fokker-Planck form:

$$\frac{\partial f}{\partial t}(\{\alpha_i\},t)=-k\sum_{\substack{ij=1\\ \mathrm{NN}}}^{N}\left(1-\frac{\partial}{\partial\alpha_j}\right)\frac{\partial}{\partial\alpha_i}(\alpha_j f(\{\alpha_i\},t))+e\sum_{i=1}^{N}\left(1-\frac{\partial}{\partial\alpha_i}\right)\frac{\partial}{\partial\alpha_i}(\alpha_i^2 f(\{\alpha_i\},t)). \qquad (6)$$

The strength of the Poisson transformation lies in two notable features: first, one obtains a simple Fokker-Planck equation (no *ad hoc* truncations), and second the equal time correlations can easily be recovered via the connection formula

$$\left\langle\!\!\left\langle\frac{(x_i)!}{(x_i-p)!}\frac{(x_j)!}{(x_j-k)!}\cdots\right\rangle\!\!\right\rangle\overset{i\neq j\neq\ldots}{=}\langle(\alpha_i)^p(\alpha_j)^k\ldots\rangle \qquad (7)$$

or

$$\langle\!\langle x_i\rangle\!\rangle=\langle\alpha_i\rangle$$
$$\langle\!\langle x_ix_j\rangle\!\rangle=\langle\alpha_i\alpha_j\rangle+\delta_{ij}\langle\alpha_j\rangle. \qquad (8)$$

Non-equal time connection formula also exist; however, the structure is more complex (Elderfield 1985a). In particular, notice that if $\rho(i)$ is the probability of finding a particle at i, and $\rho(i,j)$ the probability of finding particles at i, j, then (8) reflects in a very simple way the necessary consistency criterion

$$\rho(i,i)=\rho(i) \qquad (9)$$

for the simplest continuum limit ($\delta_{ij}\to\delta(\boldsymbol{r})$, $\langle\alpha(\boldsymbol{r})\alpha(\boldsymbol{s})\rangle\sim\exp(-\mu|\boldsymbol{r}-\boldsymbol{s}|)$, Elderfield and Vvedensky (1985b)).

Given the Fokker-Planck equation (6), it is now, despite the multiplicative nature of the underlying stochastic force, a relatively straightforward task to derive a field theoretical representation. Adopting the approach of Elderfield (1985a, b), Langouche *et al* (1979) and Elderfield and Vvedensky (1985a), which corresponds to a variant of the Martin–Siggia–Rose formalism (MSR, de Dominicis and Peliti 1978), one can show that the response/correlation functions are generated by a functional $Z(\{\hat{l}, l\})$ defined as follows:

$$Z(\{\hat{l}, l\}) \equiv \int [\mathrm{d}\alpha] \int [\mathrm{d}\hat{\alpha}]\, \delta(\alpha_1(0) - \alpha_0) \prod_{i=2}^{N} \delta(\alpha_i(0)) \exp\left(\int_0^T \mathrm{d}t(\mathscr{L} + \hat{l}\hat{\alpha} + l\alpha)\right). \tag{10}$$

The Lagrangian $\mathscr{L}$ takes the form (no truncation)

$$\begin{aligned} \mathscr{L} &= \mathrm{i} \sum_{k=1}^{N} \hat{\alpha}_k \left(\frac{\partial \alpha_k}{\partial t} - \sum_{j=1}^{N} (D_{kj}\alpha_j - \delta_{jk} e \alpha_k^2)(1 + \mathrm{i}\hat{\alpha}_j)\right) \\ D_{ij} &= \begin{cases} k & i, j \text{ NN} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{11}$$

and as usual one obtains the correlation functions through the relations

$$\begin{aligned} \langle \alpha_i(t) \rangle &= \left.\frac{\partial Z}{\partial l_i(t)}\right|_{l=\hat{l}=0} \\ \langle \alpha_i(t) \alpha_j(t') \rangle &= \left.\frac{\partial^2 Z}{\partial l_i(t)\, \partial l_j(t')}\right|_{l=\hat{l}=0} \end{aligned} \tag{12}$$

with their natural generalisations, provided $T > t, t', \ldots$, the observation times. Associated with causality and normalisation there are fundamental identities of the typical form

$$\begin{aligned} &\langle \hat{\alpha}_i(t) \rangle = 0 \\ &\langle \hat{\alpha}_i(t) \alpha_j(t') \rangle = 0 \qquad t \geqslant t'. \end{aligned} \tag{13}$$

Readers familiar with the MSR formalism will observe that there is no 'Jacobian' factor in (11), a point discussed by many authors (Langouche *et al* 1979, Leschke and Schmutz 1976). Only recently we have shown that for multiplicative stochastic forces only the chosen operator ordering/temporal discretisation is consistent with the given Fokker–Planck description† (Elderfield 1985b). Notice also that we have modified the initial condition (4), preferring instead

$$P(\{x_i\}, t) = \frac{\mathrm{e}^{-\alpha_0} (\alpha_0)^{x_1}}{(x_1)!} \left(\prod_{i=2}^{N} \delta(x_i)\right). \tag{14}$$

It is clear, however, that the self-correction mechanism (2) should quickly eliminate any discrepancy.

Given (11), a natural continuum limit presents itself. Fourier transforming the nearest-neighbour coupling D_{ij}, one has in d spatial dimensions

$$D_{ij} \to k \sum_{r=1}^{d} \cos(q_r a) \tag{15}$$

so that large distance physics, $qa \ll 1$, where a is the lattice spacing, is controlled by an effective Lagrangian of the form

$$\mathscr{L}_{\mathrm{Eden}}(\{\hat{\beta}, \beta\}) = \mathrm{i} \int \mathrm{d}\boldsymbol{r}^d \left[\hat{\beta}(\boldsymbol{r})\left(\frac{\partial \beta(\boldsymbol{r})}{\partial t} - \{\tilde{k}[(\tilde{a}\nabla)^2 + 1]\beta(\boldsymbol{r}) - \tilde{e}\beta^2(\boldsymbol{r})\}(1 + \mathrm{i}\hat{\beta}(\boldsymbol{r}))\right)\right]. \tag{16}$$

† We choose the ordering for which the path integral exhibits the most natural 'integration by parts' property.

Here we have introduced densities $\beta(\boldsymbol{r})=\alpha_i/a^d$, $\hat{\beta}(r)=\hat{\alpha}_i$ and effective couplings

$$\begin{aligned}\tilde{a}&=\frac{1}{\sqrt{2d}}a\\ \tilde{k}&=kd\\ \tilde{e}&=ea^d.\end{aligned}\tag{17}$$

Our Lagrangian (16) should be compared with that of Parisi and Zhang (1985), who suggest the following Reggeon-like Lagrangian:

$$\mathscr{L}_{\text{Eden,PZ}}(\{\hat{\phi},\phi\})=\mathrm{i}\int \mathrm{d}\boldsymbol{r}^d\left[\hat{\phi}(\boldsymbol{r})\left(\frac{\partial\phi(\boldsymbol{r})}{\partial t}-\{\tilde{k}[(\tilde{a}\nabla)^2+1]\phi(\boldsymbol{r})-\tilde{e}\phi^2(\boldsymbol{r})\}\right)-\mathrm{i}\tilde{e}\hat{\phi}^2(\boldsymbol{r})\phi(\boldsymbol{r})\right]\tag{18}$$

where the nonlinear coupling is chosen phenomenologically to give a bias against multiple occupancy.

At the mean level of approximation, the models (16) and (18) are qualitatively similar. In our case one obtains a relation of the form

$$0=\frac{\partial\mathscr{L}_{\text{Eden}}}{\partial\hat{\beta}}=\frac{\partial\beta}{\partial t}(\boldsymbol{r},t)-\tilde{k}[(\tilde{a}\nabla)^2+1]\beta(\boldsymbol{r},t)+\tilde{e}\beta^2(\boldsymbol{r},t)\tag{19}$$

since normalisation ensures $\langle\hat{\beta}\rangle=0$ (13) and then the density $\rho=x_i/a^d$ follows from the connection formula (8). Explicitly one finds that the density ρ satisfies the equation of motion

$$\frac{\partial\rho}{\partial t}(\boldsymbol{r},t)=\tilde{k}[(\tilde{a}\nabla)^2+1]\rho(\boldsymbol{r},t)-\tilde{e}\rho^2(\boldsymbol{r},t).\tag{20}$$

Given the initial conditions (4) (or (14)), one finds that the cluster grows in the first instance via

$$\rho(\boldsymbol{r},t)\sim\left(\frac{1}{\tilde{k}t}\right)^{d/2}\exp\left[\tilde{k}t-\left(\frac{r}{\tilde{a}}\right)^2\frac{1}{4\tilde{k}t}\right]\tag{21}$$

until the nonlinearity evident in (20) becomes important. In high dimensions one would not expect saturation to occur, so (21) implies

$$\begin{aligned}M&\equiv\int \mathrm{d}\boldsymbol{r}^d\langle\rho(\boldsymbol{r},t)\rangle\sim \mathrm{e}^t t^{d/2}\\ \langle R^2\rangle&\equiv\int \mathrm{d}\boldsymbol{r}^d\frac{r^2\langle\rho(\boldsymbol{r},t)\rangle}{M}\sim t\end{aligned}\tag{22}$$

whence for clusters of average mass M, the mean square size $\langle R^2\rangle$ satisfies

$$\langle R^2\rangle\cong\ln M.\tag{23}$$

in agreement with Parisi and Zhang (1984, 1985).

On the other hand, when one studies the fluctuations or corrections to the mean-field (or deterministic) growth equation (20), there are important differences. Consider for example the Langevin equations associated with (16) and (18):

$$\frac{\partial\beta}{\partial t}=\tilde{k}[(\tilde{a}\nabla)^2+1]\beta(\boldsymbol{r})-\tilde{e}\beta^2(\boldsymbol{r})+\mathrm{i}\int \mathrm{d}\boldsymbol{r}'^d\{2\tilde{k}[(\tilde{a}\nabla)^2+1]\beta-\tilde{e}\beta^2\}^{1/2}_{\boldsymbol{r}\boldsymbol{r}'}\xi(\boldsymbol{r}')\tag{24}$$

$$\frac{\partial\phi}{\partial t}=\tilde{k}[(\tilde{a}\nabla)^2+1]\phi(\boldsymbol{r})-\tilde{e}\phi^2(\boldsymbol{r})+\mathrm{i}(2\tilde{e}\phi(\boldsymbol{r}))^{1/2}\xi(\boldsymbol{r}). \tag{25}$$

Here the Ito prescription for the noise $\xi(\boldsymbol{r}, t)$ is to be understood (Gardiner 1983). Three regimes are evident: first, outside of the growing cluster the multiplicative nature of the stochasticity ensures that the fluctuations vanish, then as one enters the cluster they grow in strength and then for our model they subside again when saturation occurs. In direct contrast for the PZ description (25), which is based on a study of the flutuations about the unstable vacuum ($\langle\phi\rangle=0$) the third regime is not found due essentially to truncation errors. To recover our result PZ would have to propose a more extensive phenomenology, so our approach, firmly based on a very simple master equation (only two adjustable couplings), would appear to offer significant advantages. Moreover, our field theory in Poisson coordinates is essentially an *exact* representation of the reaction–diffusion model, so the properties of the fluctuations are fully determined.

Let us now finally consider the DLA model. In this model growth occurs by capture of a diffusing particle, and steps leading to a multiply occupied cluster site are excluded. Defining x_i to be the number of particles at site i belonging to the cluster, and d_j the number of diffusers at site j, one is led to consider the following reaction scheme:

$$D_i+X_j\xrightarrow{k}X_i+X_j \qquad i,j\ \mathrm{NN};\ p_{ij}=p_{ji} \tag{26}$$

$$2X_i\xrightarrow{e}X_i$$

$$D_i\xrightarrow{p_{ij}}D_j \tag{27}$$

$$D_i+X_i\xrightarrow{l}X_i.$$

Here the first step describes the growth process in which a diffuser D_i is captured by the cluster, and we again have a correction reaction to avoid multiple occupation. The second describes the diffusion of $\{D_i\}$ and must be supplemented by a source on some large sphere centred on the seeded site in order to ensure that the growth continues to mature. Lastly, the third process is included to hinder the diffusion of the $\{D_i\}$ into the interior of the cluster, so that the growth can be diffusion limited for any values of $k>0$. Associated with this reaction scheme is a master equation:

$$\begin{aligned}\frac{\partial}{\partial t}&P(\{x_i\},\{d_i\},t)\\ &=k\sum_{\substack{i,j=1\\ \mathrm{NN}}}^{N}(d_j+1)x_iP(x_1,\ldots,x_j-1,\ldots,x_N,d_1,\ldots,d_i+1,\ldots,d_N,t)\\ &\quad-d_jx_iP(\{x_i\},\{d_i\},t)\\ &\quad+e\sum_{i=1}^{N}x_i(x_i+1)P(x_1,\ldots,x_i+1,\ldots,x_N,\{d_i\},t)\\ &\quad-x_i(x_i-1)P(\{x_i\},\{d_i\},t)\\ &\quad+\sum_{\substack{i,j=1\\ \mathrm{NN}}}p_{ij}[(d_i+1)P(\{x_i\},d_1,\ldots,d_i+1,\ldots,d_j-1,\ldots,d_N,t)\end{aligned}$$

$$
-d_iP(\{x_i\},\{d_i\},t)]
+l\sum_{i=1}^{N}(d_i+1)x_iP(\{x_i\},d_1,\ldots,d_i+1,\ldots,d_N,t)-d_ix_iP(\{x_i\},\{d_i\},t)
\tag{28}
$$

where $P(x_1,\ldots,x_N,d_1,\ldots,d_N,t)$ is the probability of finding $\{x_i\}$ cluster particles and $\{d_i\}$ diffusers at time t. At $t=0$ we introduce a seed at the origin and throughout the process we supply a flux of diffusers. Adopting a Poisson transform description by introducing a quasi-probability via

$$
P(\{x_i\},\{d_i\},t)\equiv\int\prod_{i=1}^{N}\mathrm{d}\alpha_i\,\mathrm{d}\phi_i\left(\frac{\mathrm{e}^{-d_i}(\alpha_i)^{x_i}}{(x_i)!}\right)\left(\frac{\mathrm{e}^{-d_i}(\phi_i)^{d_i}}{(d_i)!}\right)f(\{\alpha_j\},\{\phi_j\},t)
\tag{29}
$$

leads to the Fokker-Planck equation

$$
\begin{aligned}
\frac{\partial f}{\partial t}(\{\alpha_i\},\{\phi_i\},t)
&=k\sum_{\substack{i,j=1\\ \mathrm{NN}}}^{N}\left(\frac{\partial}{\partial\phi_j}-\frac{\partial}{\partial\alpha_j}\right)\left(1-\frac{\partial}{\partial\alpha_i}\right)(\phi_j\alpha_i f(\{\alpha_i\},\{\phi_i\},t))\\
&\quad+e\sum_{i=1}^{N}\left(1-\frac{\partial}{\partial\alpha_i}\right)\frac{\partial}{\partial\alpha_i}(\alpha_i^2 f(\{\alpha_i\},\{\phi_i\},t))\\
&\quad-\sum_{\substack{i,j=1\\ \mathrm{NN}}}^{N}p_{ij}\frac{\partial}{\partial\phi_j}[\phi_i-\phi_j f(\{\alpha_i\},\{\phi_i\},t)]+l\sum_{i=1}^{N}\frac{\partial}{\partial\phi_i}\left(1-\frac{\partial}{\partial\alpha_i}\right)\\
&\quad\times(\alpha_i\phi_i f(\{\alpha_i\},\{\phi_i\},t))
\end{aligned}
\tag{30}
$$

and thence to a field theoretical representation with a Lagrangian of the form

$$
\begin{aligned}
\mathscr{L}_{\mathrm{DLA}}(\{\hat{\alpha},\alpha,\hat{\phi},\phi\})
&=\mathrm{i}\sum_{k=1}^{N}\hat{\alpha}_k\left(\frac{\partial\alpha_k}{\partial t}-\sum_{j=1}^{N}(D_{kj}\phi_k\alpha_j-\delta_{kj}e\alpha_j^2)(1+\mathrm{i}\hat{\alpha}_j)\right)\\
&\quad+\mathrm{i}\sum_{k=1}^{N}\hat{\phi}_k\left(\frac{\partial\phi_k}{\partial t}+\sum_{j=1}^{N}[D_{kj}\phi_k\alpha_j(1+\mathrm{i}\hat{\alpha}_j)-C_{kj}\phi_j]\right)\\
&\quad+\mathrm{i}\sum_{i}^{N}l\hat{\phi}_i(1+i\hat{\alpha}_i)\alpha_i\phi_i
\end{aligned}
$$

where

$$
D_{ij}=\begin{cases}k & i,j\ \mathrm{NN}\\ 0 & \text{otherwise}\end{cases}
\qquad
C_{ij}=\left(\frac{p}{k}\right)\left(D_{ij}-\delta_{ij}\sum_{r}^{N}D_{ir}\right).
\tag{31}
$$

Here $\hat{\alpha}$, α describe the growing cluster and $\hat{\phi}$, ϕ the diffusers. As one might have expected, DLA is rather more complicated than the Eden model (11). Taking the simplest continuum limit by taking the leading non-local terms as before, we obtain the effective Lagrangian

$$\begin{aligned}
&\mathscr{L}_{\mathrm{DLA}}(\{\hat{\beta}, \beta, \hat{\theta}, \theta\}) \\
&\quad = \mathrm{i}\int \mathrm{d}\boldsymbol{r}^d \bigg[\hat{\beta}(\boldsymbol{r})\bigg(\frac{\partial}{\partial t}\beta(\boldsymbol{r}) - \{\tilde{k}\theta(\boldsymbol{r})[(\tilde{a}\nabla)^2+1]\beta(\boldsymbol{r}) - \tilde{e}\beta^2(\boldsymbol{r})\}(1+\mathrm{i}\hat{\beta}(\boldsymbol{r}))\bigg) \\
&\qquad + \hat{\theta}(\boldsymbol{r})\bigg(\frac{\partial}{\partial t}\theta(\boldsymbol{r}) + \tilde{k}\theta(\boldsymbol{r})[(\tilde{a}\nabla)^2+1]\beta(\boldsymbol{r})(1+\mathrm{i}\hat{\beta}(\boldsymbol{r})) - \tilde{p}(\tilde{a}\nabla)^2\theta(\boldsymbol{r})\bigg) \\
&\qquad + \tilde{e}\hat{\theta}(\boldsymbol{r})(1+\mathrm{i}\hat{\beta}(\boldsymbol{r}))\theta(\boldsymbol{r})\beta(\boldsymbol{r})\bigg]. \qquad (32)
\end{aligned}$$

We have introduced densities $\beta = \alpha_i/a^d$, $\hat{\beta} = \hat{\alpha}_i$, $\theta = \phi_i/a^d$, $\hat{\theta} = \hat{\phi}_i$ and effective couplings

$$\begin{aligned}
\tilde{a} &= \frac{1}{\sqrt{2d}}a \\
\tilde{k} &= kda^d \\
\tilde{p} &= 2pd \qquad (33) \\
\tilde{e} &= ea^d \\
\tilde{l} &= la^d.
\end{aligned}$$

Analysing (32), we first observe that at the mean-field level

$$\frac{\partial\rho}{\partial t}(\boldsymbol{r}, t) = \tilde{k}d(\boldsymbol{r}, t)[(\tilde{a}\nabla)^2+1]\rho(\boldsymbol{r}, t) - \tilde{e}\rho^2(\boldsymbol{r}, t) \qquad (34)$$

$$\frac{\partial d}{\partial t}(\boldsymbol{r}, t) = -\tilde{k}d(\boldsymbol{r}, t)[(\tilde{a}\nabla)^2+1]\rho(\boldsymbol{r}, t) + \tilde{p}(\tilde{a}\nabla)^2 d(\boldsymbol{r}, t) \qquad (35)$$

where $\rho(\boldsymbol{r}, t)$, $d(\boldsymbol{r}, t)$ are the densities of the cluster particles and the diffusers, respectively. By contrast PZ suggest the form

$$\frac{\partial p}{\partial t}(\boldsymbol{r}, t) = \tilde{k}p(\boldsymbol{r}, t)[(\tilde{a}\nabla)^2+1]p(\boldsymbol{r}, t) \qquad (36)$$

$$0 = \tilde{k}p(\boldsymbol{r}, t)[(\tilde{a}\nabla)^2+1]p(\boldsymbol{r}, t) - \tilde{p}(\tilde{a}\nabla)^2 p(\boldsymbol{r}, t) \qquad (37)$$

where $p(\boldsymbol{r})$ is the probability of a diffuser colliding with the existing cluster. To recover this physically appealing form we must choose boundary conditions such that $d(\boldsymbol{r}) = 0$ and drop both the error correction interaction $(2X_i \to X_i)$ and diffusion limiting reaction $(D_i + X_i \to X_i)$. It is therefore quite possible that the predictions of (33) and (34) will be quite different, even at the mean-field level. To make contact with the non-local field theory proposed by PZ we observe that the field $\hat{\theta}$ in (32) can be integrated exactly out of the generating function, to obtain a delta function whose argument is the deterministic equation (35). One is therefore led to a non-local field theory of the form

$$\begin{aligned}
&\tilde{\mathscr{L}}_{\mathrm{DLA}}(\{\hat{\beta}, \beta\}) \\
&\quad = \mathrm{i}\int \mathrm{d}\boldsymbol{r}^d \bigg[\hat{\beta}(\boldsymbol{r})\bigg(\frac{\partial}{\partial t}\beta(\boldsymbol{r}) - \{\tilde{k}\theta(\boldsymbol{r}, \{\beta\})[(\tilde{a}\nabla^2)+1]\beta(\boldsymbol{r}) - \tilde{e}\beta^2(\boldsymbol{r})\} \\
&\qquad \times (1+\mathrm{i}\hat{\beta}(\boldsymbol{r}))\bigg)\bigg] \qquad (38)
\end{aligned}$$

where $d(\boldsymbol{r}, \{\beta\})$ satisfies (35) ($\beta = \rho$) with the appropriate boundary conditions. This form can then be usefully compared with that of PZ, who propose

$$\tilde{\mathscr{L}}_{\mathrm{DLA,PZ}}(\{\hat{\phi}, \phi\}) = \mathrm{i} \int \mathrm{d}\boldsymbol{r}^d \left[\hat{\phi}(\boldsymbol{r}) \left(\frac{\partial \phi}{\partial t}(\boldsymbol{r}) - \tilde{k} p(\boldsymbol{r}, \{\phi\})[(\tilde{a}\nabla)^2 + 1]\phi(\boldsymbol{r}) \right) - \mathrm{i}\hat{\phi}^2(\boldsymbol{r})\phi(\boldsymbol{r}) \right] \tag{39}$$

where $p(\boldsymbol{r}, \{\phi\})$ satisfies (37) ($\phi = \rho$). Again, as in the Eden model, we see that the fluctuations described by (38) and (39) are rather different in character.

To conclude, we have shown how field theories can be derived for the Eden model and DLA using simple reaction-diffusion master equations. Our approach using Poisson techniques is exact, so that the stochastic element of these models is determined with no ambiguity. The phenomenological equations of Parisi and Zhang (1985) would seem to be correct only in mean-field theory or in the vicinity of the 'unstable vacuum'. In regard to the general philosophy of using reaction-diffusion models we note that for example in the Eden model the essential error correction reaction ($2X \to X$) leads at time $t > 0$ to a distriution of cluster masses m in direct contrast to the original Eden model (Eden 1961) for which $t = M$. To compare, for example, the mean square cluster size we must first eliminate t in terms of M, a procedure which we *assume* leads to universal forms for the mass dependence. In support of this assertion we note that in this way we do recover the known results for the infinite dimensional Eden model (see (19) *et seq*).

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